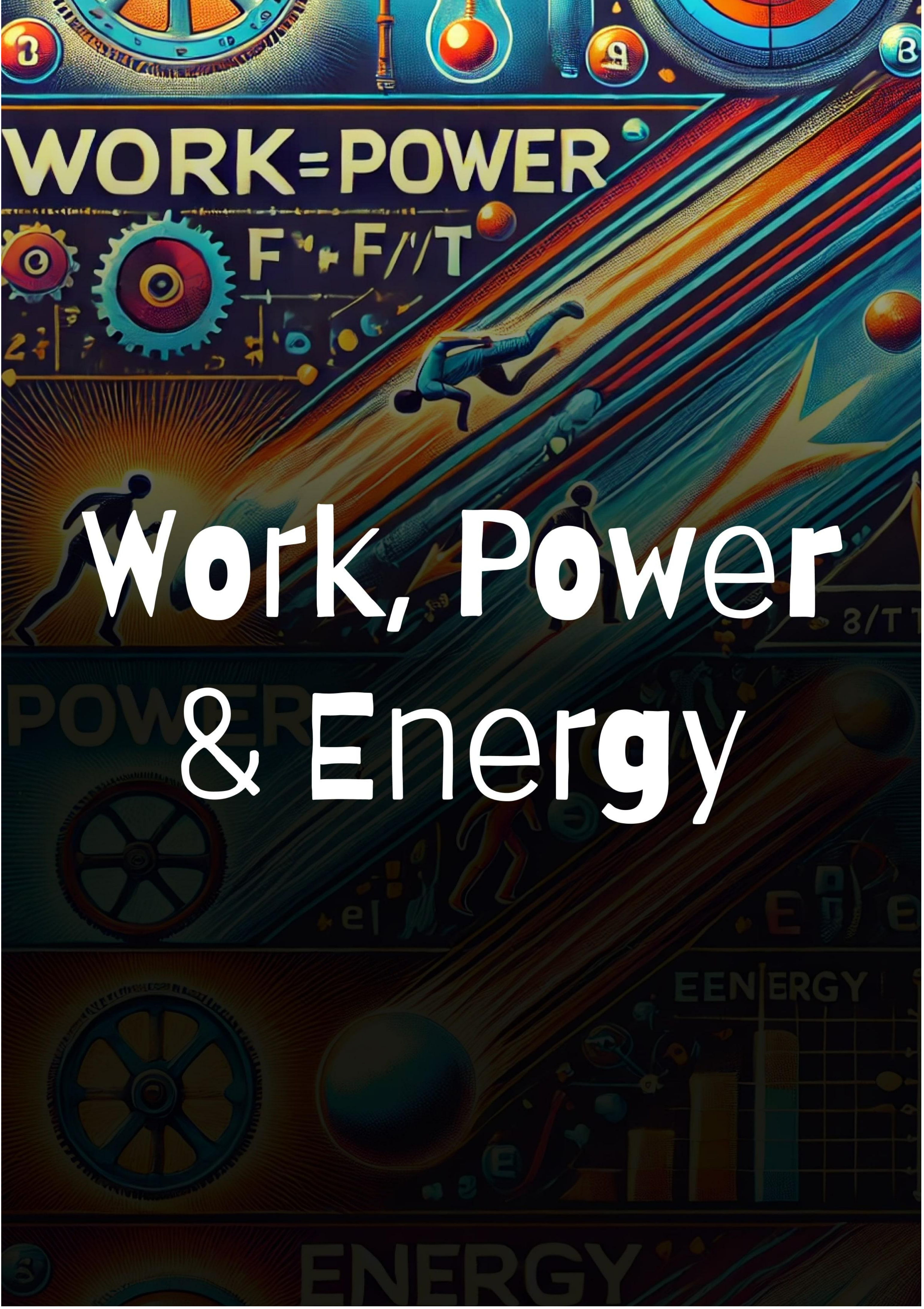


Work, Power & Energy

Work, Power & Energy

WORK

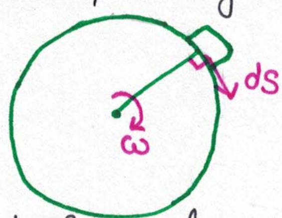
If a force $\vec{F}$ displaces a particle through a small displacement $d\vec{s}$ then the small amount of work done in process is given as

$$dw = \vec{F} \cdot d\vec{s}$$

NOTE ds is the displacement of point of application of force.

→ If displacement is perpendicular to the force, work done is zero.

For eg:- In the shown case, string tension does zero work.



From the elemental work formula, we can calculate work for a large displacement by integration.

$$w = \int \vec{F} \cdot d\vec{s}$$

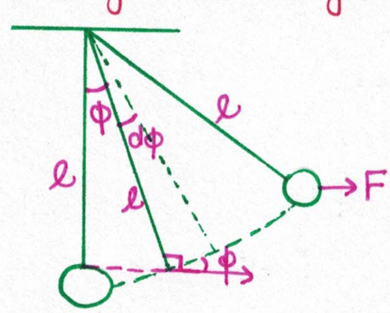
In case, we have a constant force the above formula simplifies as follows

$$w = \vec{F} \cdot \int ds$$

$$\boxed{w = \vec{F} \cdot \vec{S}}$$

Que.) Find the work done in following

(i) moving a pendulum through an angle by applying a force F.



$$ds = l\,d\phi$$

$$\int dw = \int F l\, d\phi \cdot \cos\phi$$

$$w = Fl \int_0^\theta \cos\phi \cdot d\phi$$

$$= Fl \sin\theta$$

WORK ENERGY THEOREM (WET)

Summation of work done by all the forces (both internal as well as external) on a system is equal to change in kinetic energy of the system.

Proof:

$$K.E. = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m.\vec{v}.\vec{v}$$

$$\frac{dK_E}{dt} = \frac{1}{2}m\left[\vec{a}.\vec{v} + \vec{a}.\vec{v}\right]$$

$$= m\,\vec{a}.v$$

$$\frac{dK_E}{dt} = F.\frac{ds}{dt}$$

$$\frac{dK_E}{\cancel{dt}} = \frac{dw}{\cancel{dt}}$$

$$dw = dKE$$

$$\boxed{\Delta w = \Delta K_E}$$

Que:) A block is released on a rough fixed wedge from rest. By the time block reaches the bottom. Find

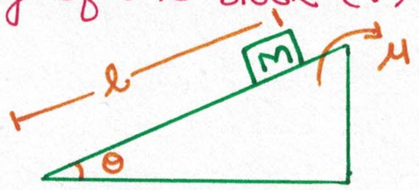
(i) Work done by normal reaction (W_N)

(ii) Work done by gravity (W_G)

(iii) Work done by frictional forces (W_f)

(iv) Total K.E. of the block (K_E)

(v) Final velocity of the block (v)



(i) $W_N = N \times l\,\cos 90^\circ = 0$

(ii) $W_f = \mu Mg\cos\theta \times l\,\cos 180^\circ = -\mu mg\cos\theta.l$

(iii) $W_G = Mg.l\,\cos(90^\circ - \theta) = Mg\,l\sin\theta$

(iv) $\Delta K_E = Mg\,l\,[\sin\theta - \mu\cos\theta] = \frac{1}{2}mv^2 - 0$

(v) $v = \sqrt{2gl(\sin\theta - \mu\cos\theta)}$

NOTE 1.) Work is dependent on frame because displacement is dependent on frame.

2.) While applying work-energy theorem in Non-inertial frames, we should also consider the work of pseudo force (applied at centre of mass).

CONSERVATIVE & NON-CONSERVATIVE FORCES

CONSERVATIVE FORCES: Those forces for which work done in a closed path is zero are called conservative force or non-decipative forces (there is no loss of mechanical energy, when a conservative force does work.)

or

For a conservative force the work done is going from one point to the other is independent of path.

Examples – 1.) Gravitational force
2.) Electrostatic force
3.) Spring force
4.) Magnetastatic force

NON-CONSERVATIVE FORCES: Those forces for which work done in a closed path is in general non-zero, are called non-conservative forces or dicipative forces. Whenever dicipative forces do some work, some energy is lost to the universe in form of heat, light & sound.

Examples - 1.) Kinetic friction
2.) Viscous drag
3.) Non-electrostatic electric field

NOTE The following forces as a pair behave like conservative forces :

1.) Normal reaction when considered as action-reaction pair.
2.) Static friction when considered as action reaction pair (as a pair, this work is not only independent of path, but it is zero for every path)
3.) Work done by tension passing over a fixed pulley when both sides of sitting are considered.

CHANGE IN POTENTIAL ENEGY

Negative of work done by a conservative force in bringing a system from one configuration to another is called change in potential energy.

POTENTIAL ENERGY

The negative of work done by a conservative force in bringing a system from certain reference configuration to the given configuration is called the potential energy of that configuration.

NOTE: The choice of reference is completely arbitrary. Hence there is no such absolute potential energy.

COROLLARIES OF WORK ENERGY THEORY

CHIME (CHANGE IN MECHANICAL ENERGY)

The summation of kinetic energy and potential energies of a system is called mechanical energy.

The work done by forces other than the conservative forces on a system is equal to change in mechanical energy of the system.

PROOF

$$ME = KE + PE$$

$$\Delta W_o + \Delta W_c = \Delta KE \quad (WET)$$

$$\Delta W_o = \Delta KE - \Delta W_c$$

$$\Delta W_o = \Delta KE + \Delta PE$$

$$\boxed{\Delta W_o = \Delta ME}$$

COME (CONSERVATION OF MECHANICAL ENERGY)

If work done by forces other than the conservative forces is zero than the mechanical energy of the system remains conserved.

Mathematically,

If $\Delta W_o = 0$, then

$$\boxed{ME = \text{const.}}$$

NOTE: There are several ways of applying COME, namely

1. Loss in P.E. = Gain in K.E.

$$KE_i + PE_i = KE_f + PE_f$$

$$PE_i - PE_f = KE_f - KE_i$$

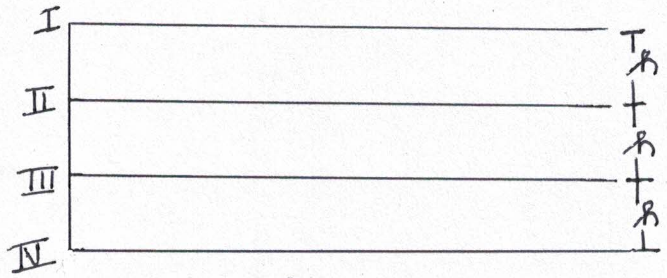
2. Loss in K.E. = Gain in PE

$$KE_i - KE_f = PE_f - PE_i$$

GRAVITATIONAL POTENTIAL ENERGY

Negative of work done by gravitational force in bringing a body from a reference level to the given level.

Eg:-



$$PE_4 = +2MgH$$

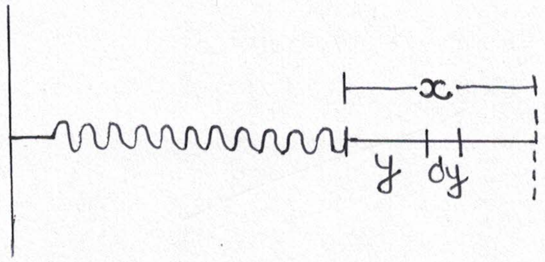
$$PE_3 = +Mgh$$

$$PE_2 = -Mgh$$

$$PE_1 = 0$$

SPRING POTENTIAL ENERGY

Negative of work done by the spring forces in bringing a spring from natural length to the given state is called spring potential energy.



$$\int dW_s = \int_0^x Ky\, dy \cos(180^\circ)$$

$$W_s = -\int_0^x Ky\, dy$$

$$W_s = -\frac{1}{2}Kx^2$$

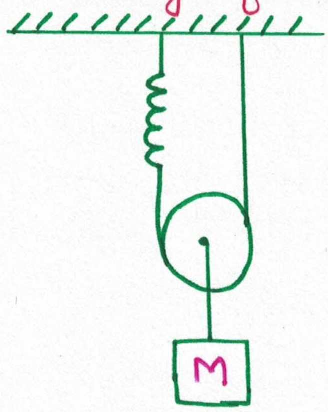
$$PE = -W_s$$

$$\boxed{PE = \frac{1}{2}Kx^2}$$

Que. The shown system is released from rest with spring initially at natural length.

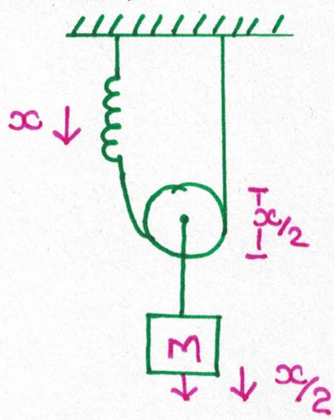
(i) What is the max. displacement of box in subsequent motion.

(ii) What is max. velocity of block :



Pulley is ideal (mass less and friction less)

(i) Loss in P.E. = Gain in K.E.



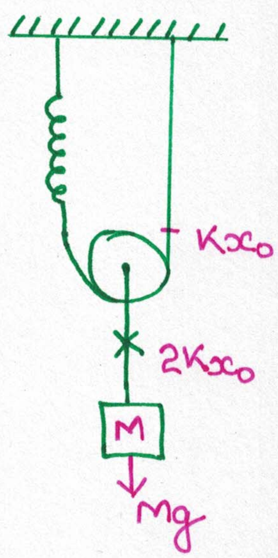
$$Mg\frac{x}{2} - \frac{1}{2}Kx^2 = 0$$

$$x = \frac{Mg}{K}$$

Max. displacement

$$\frac{x}{2} = \frac{Mg}{2K}$$

(ii) Let V_0 is the max. velocity & x_0 is the extension in the spring.



$$2Kx_0 = Mg$$

$$x_0 = \frac{Mg}{2K}$$

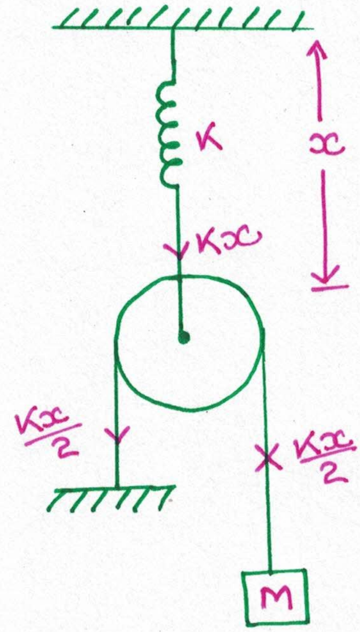
Loss in P.E. = Gain in K.E.

$$-\left(-Mg\frac{x_0}{2} + \frac{1}{2}Kx_0^2\right) = \frac{1}{2}mv_0^2$$

$$\frac{M^2g^2}{4K} - \frac{M^2g^2}{8K} = \frac{1}{2}mv_0^2$$

$$v_0 = g\sqrt{\frac{M}{4K}}$$

Que:) Repeat the previous problem for the shown fig.



(i) Loss in P.E. = Gain in K.E

$$0 - \left(-Mg\,2x + \frac{1}{2}Kx^2\right) = 0$$

$$Mg\,2\cancel{x} = \frac{1}{2}Kx^{\cancel{2}}$$

$$x = \frac{4Mg}{K}$$

$$2x = \frac{8Mg}{K}$$

(ii) $$\frac{Kx_0}{2} = Mg$$

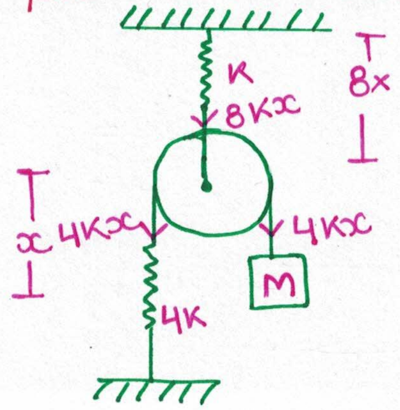
$$x = \frac{2Mg}{K}$$

$$2Mgx_0 - \frac{1}{2}Kx_0^2 = \frac{1}{2}mv_0^2$$

$$\frac{4M^2g^2}{K} - \frac{2M^2g^2}{K} = \frac{1}{2}mv_0^2$$

or $$v_0 = g\sqrt{\frac{4m}{K}} = 2g\sqrt{\frac{m}{K}}$$

Que:) For the shown system find the max. displacement of the block if the system is released from natural length condition.



Displacement of block $= 16x + x = 17x$

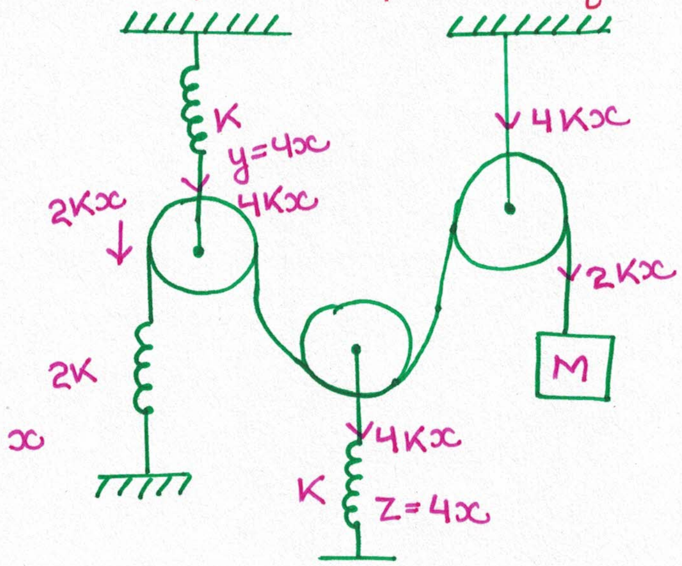
$$-\left(-17x gm + \cancel{\tfrac{1}{2}} 4Kx^2 + \tfrac{1}{\cancel{2}} K \times 34x^2\right)$$

$$17\cancel{x}gm = 34Kx^{\cancel{2}}$$

$$x = \frac{gm}{2K}$$

$$17x = \frac{17gm}{2K}$$

Que.) Repeat the previous problem for the shown case.



$$0 = -\left(-Mg\,17x + K(4x)^2 + \tfrac{1}{2} \times 2Kx^2\right)$$

$$x = \frac{Mg}{K}$$

Max. displacement $= 17x = \dfrac{17Mg}{K}$

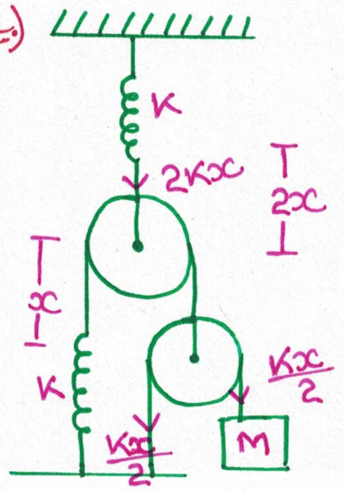
$$2y + 2z + x = 17x$$

$$x = \frac{Mg}{K}$$

Loss in P.E. = Gain in K.E.

$$\therefore\ 17x = \frac{17Mg}{K}$$

Que.)



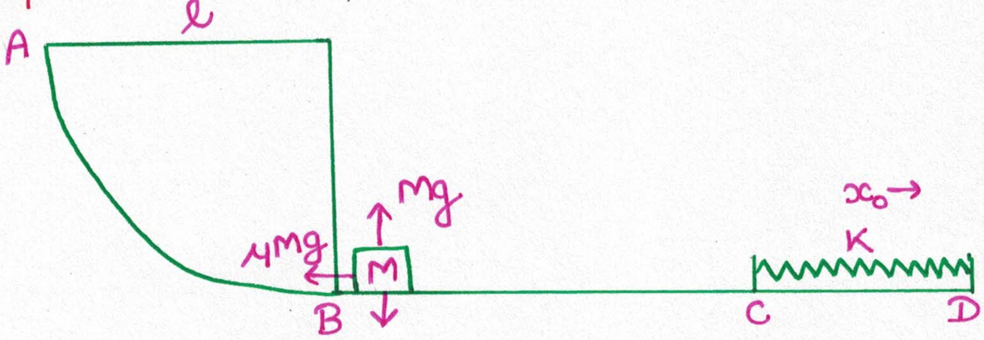
$$2x + 8x = 10x$$

Loss in P.E. = Gain in K.E.

$$x = \frac{2Mg}{5}$$

$$\therefore\ 10x = 4Mg$$

Que:) ABCD is a track in a vertical plane, having a rough section BD & a perfectly smooth section AB. Find the max. compression in the spring if a block of mass M is released from top.



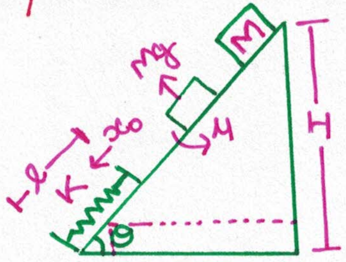
W.D by friction = change in M.E. (change in K.E.=0)

$$-\mu Mg(l+x_0) = -Mgl + \frac{1}{2}Kx_0^2 + 0$$

$$\frac{1}{2}Kx_0^2 + \mu Mg x_0 + \mu mgl - mgl = 0$$

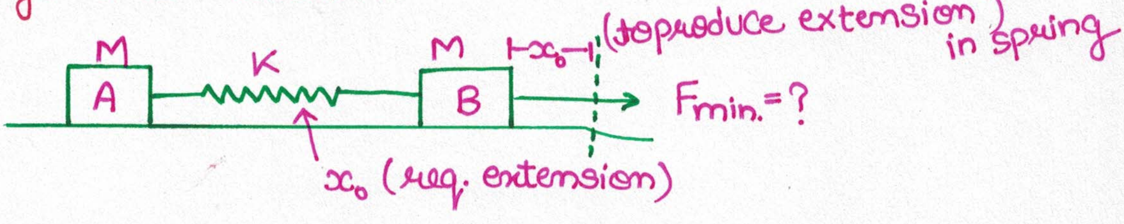
$$\frac{1}{2}Kx_0^2 + \mu Mg x_0 + mgl(\mu - 1) = 0$$

Que:) Given $\mu < \tan\theta$.
Find max. compression in the spring



$$\mu Mg\cos\theta\,(H\,\mathrm{cosec}\,\theta - l + x) = -Mg(H-(l-x)\sin\theta) + \frac{1}{2}Kx^2$$

Que:) what min. force should be applied on block B so as to just nudge the block A.

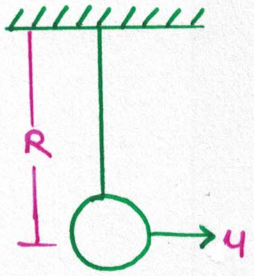


$$Kx_0 = \mu Mg$$

$$x_0 = \frac{\mu Mg}{K}$$

$F_{min.}$, $\quad x_0 - \mu Mg x_0 = \frac{1}{2}Kx_0^2$

The bob of a pendulum hanging from a string of length R is given a horizontal speed u. Find



(i) The tension in the string as a function of angle θ traversed the string.

(ii) What happens to tension as θ increases.

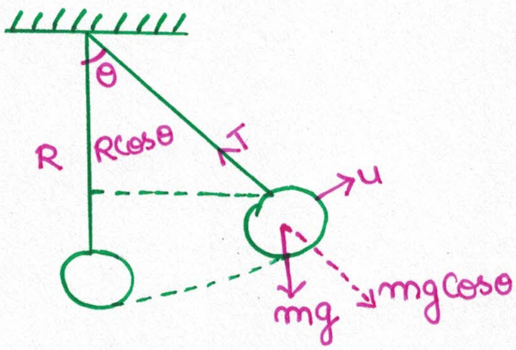
(iii) What will happen if tension becomes zero somewhere.

(iv) What should be the min. value of u to complete a vertical circle.

(v) What should be the value of u for the reversal of the string.

(vi) What will happen if the value of u is b/w that in (iv) & (v)

(Neglect air resistance)



$$T - Mg\cos\theta = \frac{Mv^2}{R} \quad \text{(equation of circular motion)} \ — ①$$

COME $\left[-MgR + \frac{1}{2}mu^2 = \frac{1}{2}mv^2 - MgR\cos\theta\right]$ —②

$$① \times R - ② \times 2$$

$$TR + 2MgR - mu^2 - MgR\cos\theta = 2MgR\cos\theta$$

(i) $T = \dfrac{Mu^2 + 3MgR\cos\theta - 2MgR}{R}$

(ii) T decreases as θ increases until $\theta = 180^\circ$

(iii) If T becomes zero, projectile motion starts. The particle deviates from circle.

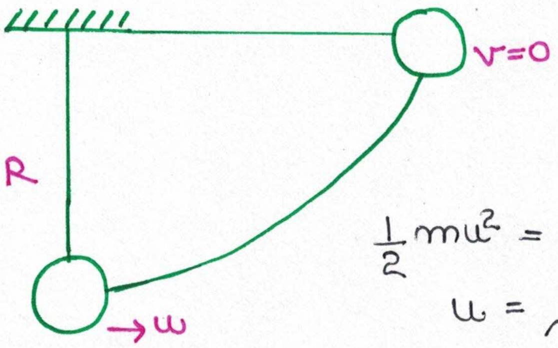
(iv) T_{min} is at $\theta = 180^\circ$. Therefore $T_{min.} > 0$ so that it completes a vertical circle.

$$Mu^2 + 3MgR\cos\theta - 2MgR > 0$$

$$Mu^2 - 3MgR - 2MgR > 0$$

$$5MgR < Mu^2$$
$$u > \sqrt{5gR}$$

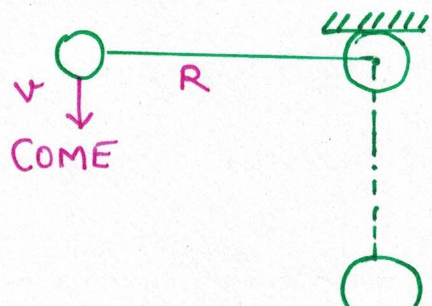
(v)



$$\frac{1}{2}mu^2 = mgR$$
$$u = \sqrt{2gR}$$

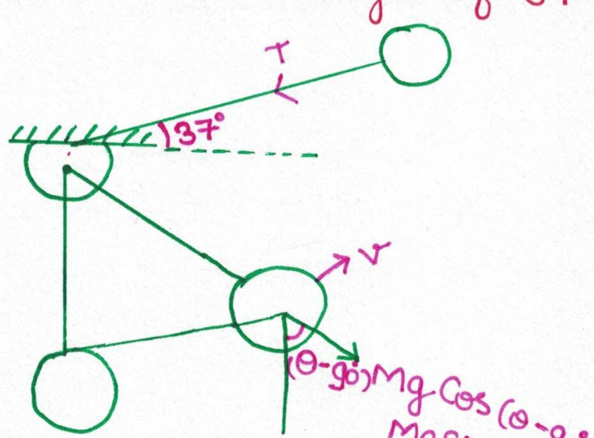
(vi) If $\sqrt{2gR} < u < \sqrt{5gR}$, string will becomes slack somewhere above the horizontal level & projectile motion will start.

Que) what should be $v_{min.}$ for the shown bob to complete the full circle.



$$\frac{1}{2}Mv^2 - 0 = \frac{1}{2}M(\sqrt{5gR})^2 - MgR$$
$$v = \sqrt{3gR}$$

Que.) For the bob shown in previous ques., what should be the min. velocity given at the bottom most point so that the string becomes slack at an angle of 37° above horizontal.



$$T - Mg\,\text{Sin}\theta = M \times \frac{v^2}{R}$$
$$g\,\text{Sin}\theta \times R = v^2$$

$$v = \sqrt{\frac{3}{5}gR}$$

COME $\quad \cancel{m}g(R + R\sin 37^\circ) = \frac{1}{2}\cancel{m}u^2 - \frac{1}{2}\cancel{m}v^2$

$$u = \sqrt{\frac{19}{5}gR}$$

Que.) A bob is released from horizontal position so that when string becomes vertical it gets tangled in a nail as shown. Find the min. value of K so that the bob executes a circle around the nail.

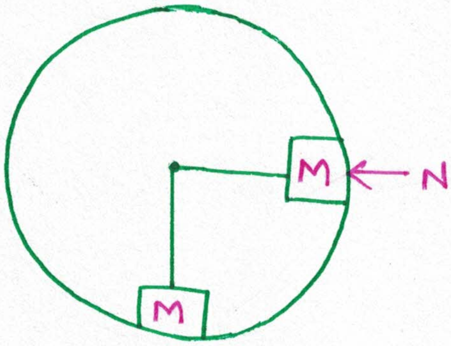
$$\sqrt{2gR} > \sqrt{5g(R - KR)}$$

$$\sqrt{1-K} < \sqrt{\frac{2}{5}}$$

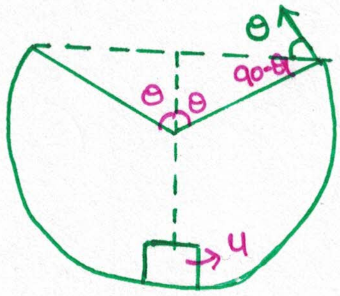
$$K > \frac{3}{5}$$

MOTION IN A VERTICAL CIRCLE INSIDE A SMOOTH HOOP

The structure of this problem is identical to that of a bob tied with the string except that, here tension is replaced by the normal reaction. Hence, all the results of the previous situation apply to this situation as well.



Que.) An arc of angle 2θ is missing from a smooth vertical hoop as shown. Find the speed required at the bottom so that the cyclic motion of the block continues perpetually.



$$2R\sin\theta = \frac{u^2 \sin 2\theta}{g}$$

$$\frac{Rg}{\cos\theta} = v^2$$

COME $\quad \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mg(R + R\cos\theta)$

Useless but valid eqn?

$$Mg\cos\theta + N = \frac{Mv^2}{R}$$

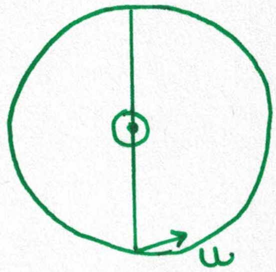
VERTICAL CIRCLE FOR A BOB ATTACHED TO A MASSLESS STICK

COME

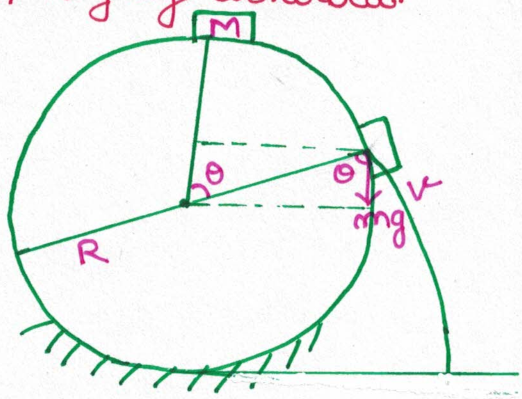
For u_{min}.

$$\frac{1}{2}mu^2 = 2MgR$$

$$u = \sqrt{4gR}$$



Que:) A block rests on top of a fixed smooth sphere of radius R. At what angle from the vertical will the block leave contact from the sphere, if slightly disturbed.



COME

$$\frac{1}{2}mv^2 = Mg(R - R\cos\theta)$$

N_C

$$\cancel{m}g.\cos\theta = \frac{\cancel{m}v^2}{R}$$

$$v^2 = Rg.\cos\theta$$

$$\frac{1}{2}\cancel{m} \times Rg\cos\theta = \cancel{M}g.(R - R\cos\theta)$$

$$\cancel{R}g\cos\theta = 2g\cancel{R} - 2\cancel{R}g.\cos\theta$$

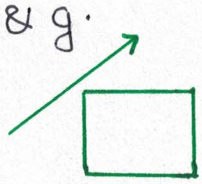
$$\cos\theta = \frac{2}{3}$$

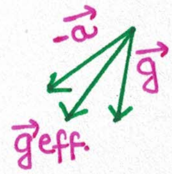
$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

PSEUDO GRAVITY

In accelerated vessel, in addition to the usual gravity, we can imagine an additional acceleration due to gravity in the direction opposite to the vessel.

For instance, for the shown vessel, the effective gravity is resultant of $-a$ & g.





PARTIAL DERIVATIVES

Let H be height of a hill, which varies as a function of two coordinates x & y in the base plane of the hill.

Then the symbol $\frac{\partial H}{\partial x}$ is called **partial derivative** of height with respect to x-coordinate.

While calculating partial derivative w.r.t. x, y is treated as a constant. Similarly the symbol $\partial H/\partial y$ is its partial derivative w.r.t. y. For calculating this, x is treated as a constant.

Que.) Height of a hill is given by $(x^2+y^2+4+3xy^2 = H)$

$$\frac{\partial H}{\partial x} \ \& \ \frac{\partial H}{\partial y} = ?$$

$$\frac{\partial H}{\partial x} = 2x + 3y^2$$

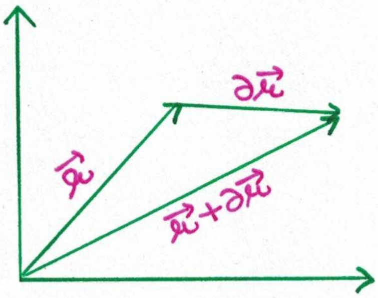
$$\frac{\partial H}{\partial y} = 2y + 6xy$$

Que.) $H = x \sin y + xy \sin^2(x) + xy^3 + 5$

$$\frac{\partial H}{\partial x} = \sin y + y^3 + y\left[x \sin 2x + \sin^2 x\right]$$

$$\frac{\partial H}{\partial y} = x \cos y + x \sin^2 x + 3xy^2$$

RELATION B/W POTENTIAL ENERGY & CONSERVATIVE FORCE



$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\partial\vec{r} = \partial x\hat{i} + \partial y\hat{j} + \partial z\hat{k}$$

$$\partial u = -\vec{F}.d\vec{r}$$

(change in potential energy)

Suppose we move along x keeping y & z Constant

$$\partial y = \partial z = 0$$

$$\partial U = -F_x.\partial x$$

$$F_x = -\frac{\partial U}{\partial x},$$

Similarly $F_y = -\dfrac{\partial U}{\partial y}$

$$F_z = -\frac{\partial U}{\partial z}$$

$$\boxed{\therefore \quad \vec{F_r} = -\frac{\partial U}{\partial r}.\hat{r}}$$

Que:) Potential energy in a region of space is given by $U = x^2+y^2+2x^2y$

Find (i) PE when the particle is at point (1,2)

(ii) Force on the particle at a general x,y

(iii) Evaluate this force at (2,3)

(i) $U = 1+4+4 = 9J$

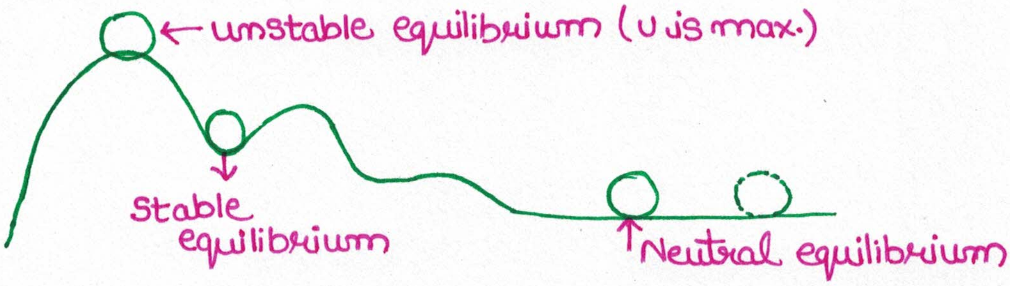
(ii) $\vec{F} = -(2x+4xy)\hat{i} - (2y+2x^2)\hat{j}$

(iii) $\vec{F}_{(2,3)} = (-4-24)\hat{i} - (6+8)\hat{j}$

$= -28\hat{i} - 14\hat{j}$

$= -28\hat{i} - 14\hat{j}$

POTENTIAL ENERGY & EQUILIBRIUM



UNSTABLE EQUILIBRIUM (U is max.)

$$\frac{\partial U}{\partial x} = 0 \quad , \quad \frac{\partial^2 U}{\partial x^2} < 0$$

STABLE EQUILIBRIUM (U is min.)

$$\frac{\partial U}{\partial x} = 0 \quad , \quad \frac{\partial^2 U}{\partial x^2} > 0$$

NOTE: For equilibrium in radially symmetric fields, $\frac{\partial U}{\partial r} = 0$. If $\frac{\partial^2 U}{\partial r^2} > 0$, it is stable equilibrium and if $\frac{\partial^2 U}{\partial r^2} < 0$, it is unstable equilibrium.

Que.) A certain potential energy field is given as $(ar^6 - br^4)$ where r is distance from the origin.

Find (i) point of equilibrium.

(ii) Comment on the nature of equilibrium.

(a & b are +ve constants.)

$$\frac{\partial U}{\partial r} = 6ar^5 - 4br^3 = 0$$

either $r = 0$

or $r = \sqrt{\frac{2b}{3a}}$

@ $r = 0$, $\frac{\partial^2 U}{\partial r^2} = 0$

we need fourth derivative

$$\frac{\partial^4 U}{\partial r^4} < 0 \quad = \text{max.}$$

118

$r=0$ is a point of unstable equilibrium.

$$\frac{\partial^2 U}{\partial r^2} = 30ar^4 - 12br^2 \qquad \left\{ r = \sqrt{\frac{2b}{3a}} \right.$$

$$= 30ar^4 - 12b \times \frac{2b}{3a}$$

$$= \frac{16b^2}{3a} > 0 \qquad \text{(minima)}$$

$\therefore$ stable equilibrium

FINDING THE P.E. FUNCTION FROM THE FORCE FUNCTION (PARTIAL INTEGRATION)

While taking partial integral w.r.t a variable, we have to add an arbitrary function of the other variables.

Que.) A 2D-dimensional force field is given as $F = -(2xy + 3y^2 + 3)\hat{i} - (x + 6xy + 2)\hat{j}$. Find corresponding P.E. function.

$$\frac{\partial U}{\partial x} = 2xy + 3y^2 + 3$$

$$U = yx^2 + 3y^2x + 3x + f(y)$$

$$\frac{\partial U}{\partial y} = x^2 + 6yx + 2$$

$$\frac{\partial U}{\partial y} = x^2 + 6xy + \frac{df}{dy}$$

$$= x^2 + 6xy + 2$$

$$\frac{df}{dy} = 2$$

$$f = 2y + C$$

$$U = x^2y + 3y^2x + 3x + 2y + C$$

Que.) Find the potential energy function if $\vec{F} = -(2x+y)\hat{i} - (x+4y)\hat{j}$

P.E. at $(1,2)$ is 0.

$$U = x^2 + xy + 2y^2 + C$$

$$1 + 2 + 8 + C = 0$$

$$C = -11$$

$$U = x^2 + xy + 2y^2 - 11$$

NOTE: If potential energy cannot be defined corresponding to a given force field then the field force is a non-conservative force.

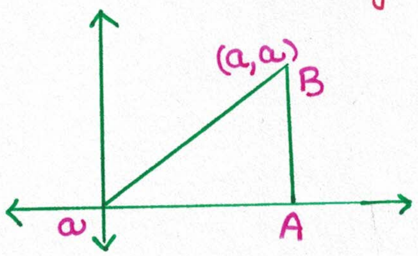
Que.) A certain force field is given by $\vec{F} = y\hat{i} - x\hat{j}$

(i) Is the field conservative

(ii) Find the work done in going from point a to B along

(a) directly OB

(b) along OAB



$$U = -xy + f(y)$$

$$\frac{\partial u}{\partial x} = -y$$

$$\frac{\partial u}{\partial y} = x = -x + \frac{df}{dy}$$

$$\frac{df}{dy} = 2x$$

Contradiction

$\therefore$ Force is non-conservative.

$$\vec{F} = y\hat{i} - x\hat{j}$$

$$d\vec{s} = dx\hat{i} + dy\hat{j}$$

$$dW = \vec{F}.d\vec{s}$$

$$dW = y.dx - x.dy$$

Path $\rightarrow$ OB

$$y = x$$

$$dy = dx$$

$$dW = x.dx - x.dx = 0.dx$$

$$W = \int_0^a 0.dx = 0$$

Path $\rightarrow$ OA

$$y = 0$$

$$dy = 0$$

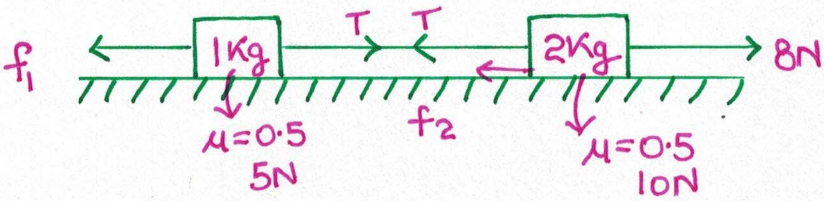
$$dW = 0dx - x \times 0 = 0.dx$$

$$W_{OA} = \int_0^a 0.dx = 0$$

Path → AB

$$x = a$$
$$dx = 0$$
$$dW = 0 - a \cdot dy$$
$$W = -\int_0^a a \cdot dy = -a^2$$
$$W_{OAB} = -a^2$$

Que.) In the shown system, find the tension & the frictional forces. Given the initial tension in the string is zero.



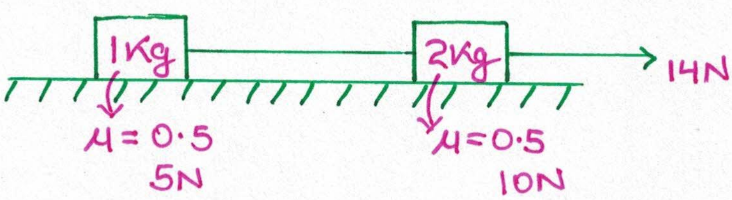
$$f_1 + f_2 = 8N$$
$$8 = T + f_2$$
$$T = f_1$$

It is a statically indeterminent problem.

NOTE: For solving statically indeterminant problem, we assume that initial tension in the string is zero. In this chain, we begin the problem from the extreme towards which, there is an overall tendency of slipping.

$\therefore f_1 = 8N$ and $f_2 = 0N$ & $T = 0N$

Que.) Solve for tension & frictional forces.



We start adjusting friction from 2Kg block

$f_1 = 10N$, $f_2 = 4N$, $T = 4N$